MATH2050C Selected Solution to Assignment 12

Section 5.6 no. 3, 4, 11.

(3) It is clear that both functions are strictly increasing everywhere. Their product $h(x) = x(x-1)$ satisfies $h(0) = h(1) = 0$ so it cannot be increasing on $[0, 1]$. Indeed, if h is increasing, it implies that h is the constant zero function which is clearly ridiculous. In general, the product of two non-negative, increasing functions is increasing.

(4) Let f and g be two positive, increasing function and let $x < y$ be two points in their domain of definition. Then,

$$(fg)(x) - (fg)(y) = f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)) \leq 0 \ ,$$

so fg is increasing.

(11) The inverse function of f, g, is defined on $[0, 1] \bigcup (2, 3]$ satisfying $g(y) = y, y \in [0, 1]$ and $g(y) = y - 1, y \in (2, 3]$. It is continuous every on its domain of definition. This example shows that the inverse of a discontinuous function could be a continuous function.

Supplementary Problems

1. (Optional) Order the rational numbers in $(0, 1)$ into a sequence $\{x_k\}$. Define a function on $(0, 1)$ by $\varphi(x) = \sum 1/2^k$ where the summation is over all indices k such that $x_k < x$. Show that

   (a) φ is strictly increasing and $\lim_{x \to 1^-} \varphi(x) = 1$.

   (b) φ is discontinuous at each x_k.

   (c) φ is continuous at each irrational number in $(0, 1)$.

Solution A sketchy proof. (a) It is obvious that φ is strictly increasing and $\lim_{x \to 1^-} \varphi(x) = 1$ since $\sum_{k=1}^{\infty} 2^{-k} = 1$.

(b) Observe that $j_\varphi(x_k) \geq 2^{-k} > 0$.

(c) Given $\varepsilon > 0$, fix a large k_0 such that $\sum_{k=k_0+1}^{\infty} 2^{-k} < \varepsilon$. Let $z \in (0, 1)$ be irrational. We can find a small δ such that $(z - \delta, z + \delta)$ does not contain any x_k with index $k \leq k_0$. Then for $x < y, x, y \in (z - \delta, z + \delta)$,

$$0 < \varphi(y) - \varphi(x) \leq \sum_{k=k_0+1} 2^{-k} < \varepsilon \ ,$$

hence φ is continuous at z.

Note This example shows how complicated a monotone function could be.